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Derivation of the upper limit of temperature from the field theory of thermodynamics

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In the present Rapid Communication we calculate the density matrix of heat conduction based on the field theory of nonequilibrium thermodynamics, which was worked out in the last 10 years. Applying these results we can discuss the existence of the maximal temperature and a possible upper limit for its value. We point out, proposing relevant physical assumptions, that this temperature could be the so-called Planck temperature [J. A. S. Lima and M. Trodden, Phys. Rev. D **53**, 4280 (1996)].

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I. INTRODUCTION

The question regarding the maximal temperature is not new at all. Moreover, if a maximal temperature exists, what happens close to that? Our aim is to find a reasonable answer to this question. We would like to emphasize that mainly the mathematics will lead our examination via relevant physical assumptions. The mathematical calculus of Hamiltonian formalism is the starting point, which has been developed and applied to describe nondissipative physical processes. Previously, we have shown that this mathematical method can be modified for dissipative processes, e.g., heat conduction. Thus, we start from the description of thermal field; we apply the methods of field theory, i.e., Hamilton's principle, Lagrangian of Fourier equation; we write the Hamilton-Jacobi equation; and we calculate the action. The propagator and the density matrix of the thermal process can be simply obtained by the usual method. These steps will bring us closer to formulating the mathematical restrictions that contain the limits of extremal behaviors of this physical system. At this point, making some assumptions about the characteristic length, we can handle this "thermal field" as a quantum field with massive thermal particles. We may call these "hotons."

There are several papers in the literature that deal with the existence of thermal particles with mass from different viewpoints. One of these is Veinik's hypothesis of elementary thermal quanta [1,2], which observed in terms of the entropy of a single photon of the blackbody radiation. Another direction in the research is to apply complex valued field variables in the Lagrangian theory of nonequilibrium processes. Anthony *et al.* [3–6] introduced the concept of the thermal excitations or thermal "thermion" field, in which quasiparticles are associated with the matter field.

II. PRELIMINARIES

A physical process can be described by the Lagrangian, i.e., all of the information of evolution of a physical process are involved in this scalar function. The time integration of the Lagrangian yields the classical action $S[b,a]$,

$$S[b,a]=\int_{t_a}^{t_b} L(\dot{q},q,t)dt, \tag{1}$$

where a belongs to the initial state at time t_a, and b is the final one at t_b. The classical Lagrangian L of the physical problem may depend on q and its time derivatives (here just $\dot{q}$) and the time t. Hamilton's principle states that the variation of action is zero for the real physical processes $\delta S=0$. The field equation of Fourier heat conduction is

$$\frac{\partial T}{\partial t}-\frac{\lambda}{c_v}\Delta T=0, \tag{2}$$

where T is the temperature, λ is the heat conductivity, and c_v is the specific heat capacity; Δ denotes the Laplace operator. We introduce a scalar, differentiable (potential) field φ [7],

$$T=-\frac{\partial\varphi}{\partial t}-\frac{\lambda}{c_v}\Delta\varphi, \tag{3}$$

which is connected to the measurable field T in this equation. The Lagrange density function of the physical problem expressed by φ is

$$L=\frac{1}{2}\left(\frac{\partial\varphi}{\partial t}\right)^2+\frac{1}{2}\frac{\lambda^2}{c_v^2}(\Delta\varphi)^2, \tag{4}$$

from which we can obtain the field equation of φ as the Euler-Lagrange equation. We can choose this quadratic Lagrangian, because in the complicated cases we can find the transformation of the variables by which we obtain this Lagrangian. On the other hand, the additional terms appearing in the Lagrangian make more difficult the mathematical calculations from the viewpoint of the construction of the canonical theory and the quantization. If we substitute the equation of definition of φ, we receive the field equation of Fourier heat conduction [7]. We can write φ in a Fourier series [8],

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$$\varphi = \sum_{k>0} \sqrt{\frac{2}{V}}(C_k \cos kx + S_k \sin kx), \tag{5}$$

where C_k and S_k are the function of time, and these will be the generalized coordinates of the system. The Lagrangian of the field can be calculated if we substitute the Fourier series of φ into Eq. (4), and then we integrate over the volume. We get the Lagrangian of the space, which depends on only the generalized coordinates and the first-order derivatives of these with respect to time,

$$L = \frac{1}{2}\sum_{k>0}\left[(\dot{C}_k^2 + \dot{S}_k^2) + \frac{\lambda^2}{c_v^2}k^4(C_k^2 + S_k^2)\right]. \tag{6}$$

The canonically conjugated quantities are the momenta $P_K^{(C)}=\dot{C}_k$ and $P_K^{(S)}=\dot{S}_k$, by which we can express the Hamiltonian of the system [8]

$$H = \sum_k \left(\frac{1}{2}P_k^{(C)^2} - \frac{1}{2}\frac{\lambda^2}{c_v^2}k^4C_k^2\right) + \sum_k \left(\frac{1}{2}P_k^{(S)^2} - \frac{1}{2}\frac{\lambda^2}{c_v^2}k^4S_k^2\right). \tag{7}$$

III. THE ACTION, THE KERNEL, AND THE DENSITY MATRIX

The Hamilton-Jacobi equation can be written, in general,

$$\frac{\partial S}{\partial t} + H\left(q_1, \ldots, q_f, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_f}\right) = 0, \tag{8}$$

where q_i is a generalized coordinate, and $\partial S/\partial q_i$ is a momentum. In our special case we can express the Hamilton-Jacobi equation of heat conduction by the Fourier coefficients as generalized coordinates,

$$\frac{\partial S}{\partial t} + \sum_k \left[\frac{1}{2}\left(\frac{\partial S}{\partial C_k}\right)^2 - \frac{1}{2}\frac{\lambda^2}{c_v^2}k^4C_k^2\right] + \sum_k \left[\frac{1}{2}\left(\frac{\partial S}{\partial S_k}\right)^2 - \frac{1}{2}\frac{\lambda^2}{c_v^2}k^4S_k^2\right] = 0. \tag{9}$$

The system is developing from the state a at time t_a to the state b at time t_b, and we suppose $t_b > t_a$. Solving this partial differential equation, and taking into account the initial and final states, the calculated action for this process is

$$S[b,a] = \sum_{k>0} \frac{(\lambda/c_v)k^2}{2\sinh[(\lambda/c_v)k^2(t_b - t_a)]} \times \left\{(C_{ka}^2 + C_{kb}^2 + S_{ka}^2 + S_{kb}^2)\cosh\left[\frac{\lambda}{c_v}k^2(t_b - t_a)\right] - 2C_{ka}C_{kb} - 2S_{ka}S_{kb}\right\}, \tag{10}$$

which is the solution of the Hamilton-Jacobi equation. In general, the kernel of a quadratic action can be written [9] (or can be calculated by the path integral method [10–12])

$$K(b,a) = \prod_{k>0} \frac{(\lambda/c_v)k^2}{2\pi i h^* \sinh[(\lambda/c_v)k^2 t]} \times \exp\left\{\frac{(\lambda/c_v)k^2 i}{2h^* \sinh[(\lambda/c_v)k^2 t]} \times\left[(C_{ka}^2 + C_{kb}^2 + S_{ka}^2 + S_{kb}^2)\cosh\left(\frac{\lambda}{c_v}k^2 t\right) - 2C_{ka}C_{kb} - 2S_{ka}S_{kb}\right]\right\}, \tag{11}$$

where we denote $t=t_b-t_a$. This propagator [13] is called a WKB propagator [9], and it is exact for those Lagrangians that contain quadratic terms, in general. It can be simply proved that this propagator is a solution of the following generalized Schrödinger-type equation:

$$-\frac{h^*}{i}\frac{\partial K}{\partial t} = -\sum_k \frac{h^{*2}}{2}\frac{\partial^2 K}{\partial C_k^2} - \sum_k \frac{h^{*2}}{2}\frac{\partial^2 K}{\partial S_k^2} - \sum_k \frac{\lambda^2 k^4}{2c_v^2}(C_k^2 + S_k^2)K, \tag{12}$$

which shows the correctness of the propagator from another side. Here, h^* denotes the unit of action given by Eq. (10) with its measure $h^* = 2\hbar/k_B$, where $\hbar$ is the Planck constant per 2π, and k_B is the Boltzmann constant. Now, we are in a position to give the density matrix of heat conduction. This can be simply achieved by going over to the propagator with imaginary time, i.e., by the substitution

$$t \to \frac{1}{i}\frac{h^*}{8\pi T}, \tag{13}$$

similar to the literature [9], $t \to \hbar/i4\pi k_B T$. Thus, we may obtain the density matrix of heat conduction

$$\rho[b,a;1/(k_BT)] = \prod_{k>0} \frac{(\lambda/c_v)k^2}{2\pi\hbar \sin\left(\frac{(\lambda/c_v)k^2\hbar}{4\pi k_B T}\right)} \times \exp\left(-\frac{(\lambda/c_v)k^2}{2\hbar \sin\left(\frac{(\lambda/c_v)k^2\hbar}{4\pi k_B T}\right)} \times\left[(C_{ka}^2 + C_{kb}^2 + S_{ka}^2 + S_{kb}^2)\cos\left(\frac{(\lambda/c_v)k^2\hbar}{4\pi k_B T}\right) - 2C_{ka}C_{kb} - 2S_{ka}S_{kb}\right]\right), \tag{14}$$

which is expressed by the generalized coordinates. The applicability of the Feynman path integral method shows and carries the construction of wave function that is related to the propagator K. This propagator is a solution of a Schrödinger-type equation. These facts may indicate the possibility of particle-wave duality [14–16] in the present case.

IV. ON THE MAXIMAL TEMPERATURE

The density matrix is expressed by the Fourier coefficients as generalized coordinates, so it is not easy to interpret the meaning of it for the first view. We exploit the property of the density matrix, namely, this must be positive for all wave numbers k. Thus, we should restrict our examination to the coefficients,

$$\frac{(\lambda/c_v)k^2}{2\pi\hbar \sin\left(\dfrac{(\lambda/c_v)k^2\hbar}{4\pi k_B T}\right)}.$$

This expression is always positive if the sine is positive in the denominator. In this way, the following nonequality holds:

$$0 < \frac{(\lambda/c_v)k^2\hbar}{4\pi k_B T} = \frac{Dk^2\hbar}{4\pi k_B T} < \pi, \tag{15}$$

by which the expected property of the density matrix is carried out. Here, we denoted the heat diffusivity $D=\lambda/c_v$. We introduce the thermal wavelength λ_T instead of the wave number $k=2\pi/\lambda_T$, and we obtain

$$\lambda_T > \sqrt{\frac{4\pi D\hbar}{k_B T}}. \tag{16}$$

What might be the minimal wavelength? What may be the physically acceptable minimum of that length where the above description of heat transfer is valid? It is not easy to answer and probably the usual heat conduction and the usual meaning of temperature do not exist at all. However, we may assume that there is a kind of energy transfer that is very similar to a heat transfer (a slow, nonrelativistic, linear process) and this can be described by a Fourier-like equation. We use the diffusivity D as a parameter that includes all of the properties of system, and it is not necessary to be expressed as the function of λ and c_v. We can calculate the maximal temperature from the above equation and we get by the minimal length $L(=\lambda_{T,min})$

$$T_{max} = \frac{D\hbar}{k_B L^2}. \tag{17}$$

The maximal temperature depends on universal coefficients h and k_B, the measurement L, and the diffusivity D. We calculate the diffusivity and then we can obtain the upper limit of temperature. The Feynman path integral quantization handles the interacting quanta as wave packets with the energy $D\hbar k^2$ [see Eq. (14)]. Let us write this thermal process with particles that are unknown at this moment. If we accept their possible existence, we can write the de Broglie wavelength λ_D of these particles with their mass m as

$$\lambda_D = \sqrt{\frac{4\pi^2\hbar^2}{mk_B T}}. \tag{18}$$

We connected the same process from different viewpoints. We obtained the thermal wavelength λ_T from our density matrix [Eq. (14)] and we write the de Broglie wavelength λ_D from the statistical physics. It seems to be obvious that these wavelengths are equal, thus, we are allowed to write $\lambda_T = \lambda_D$. We express the diffusivity

$$D = \frac{4\pi^2\hbar}{m}. \tag{19}$$

Now, we can handle the thermal particle (hoton) as a de Broglie particle by the equation

$$D\hbar k^2 = D\hbar\frac{4\pi^2}{L^2} = mc^2, \tag{20}$$

which shows that particle with mass m can appear and can carry the thermal property of heat transfer; c is the speed of light. Substituting Eqs. (19) and (20) into Eq. (17) we obtain the maximal temperature

$$T_{max} = \frac{c\hbar}{k_B L}. \tag{21}$$

One can choose a minimal value of this length for the Planck length,

$$L_p = \sqrt{\frac{\hbar\gamma}{c^3}}, \tag{22}$$

γ is the gravitational constant, by which the maximal temperature is

$$T_{max} = \sqrt{\frac{c^5\hbar}{k_B^2\gamma}}. \tag{23}$$

This is exactly the so-called Planck temperature [17]. It is known from the cosmology, and it is considered as an absolute limit for the temperature which is about 1.4×10^{32} K.

V. SUMMARY

We can conclude that the maximal temperature—which was calculated from the classical thermodynamic viewpoint as a starting point—is exactly the same as the Planck temperature, which was introduced in the cosmology. The possibility of this calculation was rooted in the existence of Hamilton-Lagrange formalism of the classical dissipative processes. This formulation allowed us to exploit the Feynman method and we could calculate the density matrix of a thermal process. This allowed us to assume the particle behavior, and in this sense we made some relevant physical assumptions to give the value of the possible maximal temperature, i.e., there is no higher physically meaningful temperature.

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